About stratification by orbit types

Julien M. P. Giacomoni

About the stratification by orbit types

Giacomoni Julien

March 12, 2014

Abstract

When we have a proper action of a Lie group on a manifold, it is well known that we get a stratification by orbit types and it is known that this stratification satifies the Whitney (b) condition. In this article we will see that this stratification satisfies the strong Verdier condition.

This article is based on an idea of David Trotman.

1 Stratification by orbit types

In this section we will recall the definitions and principal results about stratifications by orbit type. The classical references underlying what will come are [8], [14]. We will follow mostly the notations of M. Pflaum in [15] which synthesizes works by [1], [3], [5], [7], [10], [16].
Let $\mathcal{M}$ be a manifold and G a Lie group.

Definition 1. *A (left) action of G is a smooth mapping (i.e. $\mathcal{C}^\infty$) $\Phi : G \times \mathcal{M} \to \mathcal{M}$, $(g, x) \mapsto \Phi(g, x) = \Phi_g(x) = gx$ such that: $\forall g, h \in G, \forall x \in \mathcal{M}, \Phi_g(\Phi_h(x)) = \Phi_{gh}(x), \Phi_e(x) = x$, where e is the unit element of G.*

Definition 2. *A G-action $\Phi : G \times \mathcal{M} \to \mathcal{M}$ is called* proper *if the mapping $\Phi_{ext} : G \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, $(g, x) \mapsto (gx, x)$ is proper.*

With such proper actions several results are known, in particular $\mathcal{M}$ admits a G-invariant Riemannian metric. The most important result is the so called slice theorem ([8], [14]). Here it is as stated in [15]:

Theorem 1. *Let $\Phi : G \times \mathcal{M} \to \mathcal{M}$ be a proper group action, x a point of $\mathcal{M}$ and $\mathcal{V}_x = T_x\mathcal{M}/T_xGx$ the normal space to the orbit of x. Then there exists a G-equivariant diffeomorphism from a G-invariant neighborhood of the zero section of $G \times_{G_x} \mathcal{V}_x$ onto a G-invariant neighborhood of Gx such that the zero section is mapped onto Gx in a canonical way (where G_x is the isotropy group of x).*

If we denote $\mathcal{M}_{(H)}$ the set $\{x \in \mathcal{M} | G_x \sim H\}$ where G_x is the isotropy group of x and $\sim$ means "conjugate to", we get in particular that for a compact subgroup H of G each connected component of $\mathcal{M}_{(H)}$ is a submanifold of $\mathcal{M}$. The isotropy subgroups G_x are compact in the case of a proper group action. Assigning to each point $x \in \mathcal{M}$ the germ $\mathcal{S}_x$ of the set $\mathcal{M}_{(G_x)}$ we get a stratification of $\mathcal{M}$ in the sense of Mather ([13]), called stratification by orbit type.
This stratification has been studied a lot and has been also recently described in [6], [4]. This stratification is known to be Whitney (b) regular.

2 Verdier's condition

About Verdier's condition the reader may look for [19], [9], [17], [2]:

Definition 3. *Let X be a $\mathcal{C}^1$ submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset \overline{X} \setminus X$. Verdier ([19]) defines X to be (w)-regular over Y at 0 if there is a constant $C > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqslant C|x - y|$.*

(with $d(A,B) = sup\{dist(x,B)|x \in A, |x| = 1\}$).
This condition is known to be stronger than the Whitney (b) condition for subanalytic sets (Kuo has shown that condition (w) implies condition (b) in [9] and Trotman in [18] has shown that the converse is false (in the real case)). But we also have a stronger version of Verdier's condition:

Definition 4. *Let X be a $\mathcal{C}^1$ submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset \overline{X} \setminus X$. In [11] (see also [12]) Li, Kuo, Trotman and Wilson define X to be strongly Verdier regular over Y (or differentiably regular) at 0 if for all $\epsilon > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqslant \epsilon|x-y|$.*

The next theorem is an enhancement of the theorem 4.3.7 that can be found page 160 in [15], most of the notations will be conserved.

Theorem 2. *The stratification by orbit types of a G-manifold $\mathcal{M}$ with a proper action is a strong Verdier stratification.*

Proof. Suppose that $K \subsetneq H \subset G$ are two isotropy groups of $\mathcal{M}$, we have $\mathcal{M}_{(H)} < \mathcal{M}_{(K)}$. Let $y \in \mathcal{M}_{(H)}$. With the slice theorem, we can suppose that: $\mathcal{M} = G \times_H \mathcal{V} = (G \times_H \mathcal{W}) \times \mathcal{V}^H$ et $y = [(e,0)]$ where $\mathcal{V}$ is an H-slice, $\mathcal{V}^H$ is the subspace of the H-invariant vectors, and $\mathcal{W} = (\mathcal{V}^H)^\perp$ is the orthogonal space relative to the H-invariant inner product on $\mathcal{V}$. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ that of H, and $\mathfrak{m}$ the orthogonal space of $\mathfrak{h} \subset \mathfrak{g}$ related to the H-invariant inner product on $\mathfrak{g}$. By the exponential map on G we have a natural chart on an open set $\mathcal{U}$ of $\mathcal{M}$ containing y:
$\phi : \mathcal{U} \to \mathfrak{m} \times \mathcal{V}, \phi([(exp(\xi), \nu]) = (\xi, \nu), \xi \in \mathfrak{m}, \nu \in \mathcal{V}$. We have ([15] page 159):$\mathcal{M}_{(K)} = (G \times_H \mathcal{W}_{(K)}) \times \mathcal{V}^H$ et $\mathcal{M}_{(H)} = G/H \times \{0\} \times \mathcal{V}^H$ and through this chart on $\mathcal{U}$ they become parts of $\mathfrak{m} \times \mathcal{W}_{(K)} \times \mathcal{V}^H$ and $\mathfrak{m} \times \{0\} \times \mathcal{V}^H$. This chart is smooth and so we can check Verdier's condition (which is $\mathcal{C}^2$-invariant) at $\tilde{y} = (0,0,0)$ in $\phi(\mathcal{M}_{(H)} \cap \mathcal{U})$ (open set of $\mathfrak{m} \times \{0\} \times \mathcal{V}^H$), let $\tilde{x} \in \phi(\mathcal{U} \cap \mathcal{M}_{(K)})$ (open set of $\mathfrak{m} \times \mathcal{W}_{(K)} \times \mathcal{V}^H$) we have,
$T_{\tilde{y}}(\phi(\mathcal{M}_{(H)} \cap \mathcal{U})) = \mathfrak{m} \times \{0\} \times \mathcal{V}^H \subset T_{\tilde{x}}(\phi(\mathcal{U} \cap \mathcal{M}_{(K)})$, (we also have a strict inclusion because $\mathcal{W}_{(K)}$ is invariant by multiplication by a non-vanishing scalar), so we have $d(T_{\tilde{y}}(\phi(\mathcal{M}_{(H)} \cap \mathcal{U})), T_{\tilde{x}}(\phi(\mathcal{U} \cap \mathcal{M}_{(K)})) = 0$ and so strong Verdier condition holds at $\tilde{y}$ (in fact we have something even stronger). □

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[2] H. Brodersen and D. Trotman, *Whitney (b)-regularity is weaker than Kuo's ratio test for real algebraic stratifications*, Math. Scand. 45, 1979, 27-34.

[3] K. Dovermann and R. Schultz, *Equivariant surgery theories and their periodicity properties*, Lecture Notes in Mathematics, vol. 1443, Springer-Verlag, 1990.

[4] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Springer-Verlag, Heidelberg, 2000.

[5] M. Ferrarotti, *G-manifolds and stratifications*, Rend. Ist. Mat. Univ. Trieste, 26, 1994.

[6] M. J. Field, *Dynamics and symmetry*, ICP Advanced Texts in Mathematics-Vol 3, Imperial College Press, 2007.

[7] K. Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Lectore Notes in Mathematics, vol. 59, Springer-Verlag, Berlin, Heidelberg, New York, 1968.

[8] J.L. Koszul, *Sur certains groupes de transformation de Lie*, Colloque de Géométrie différentielle, Colloques du CNRS, 1953, 137-141.

[9] T. C. Kuo, *The ratio test for analytic Whitney stratifications*, Proceedings of Liverpool Symposium I, Springer Verlag Lecture Notes,1971, p192.

[10] M. Lesch, *Die Struktur von Orbiträumen*, Vortragsausarbeitungen eines Vortrags bei der Max-Planck-Arbeitsgruppe Potsdam, September 1992.

[11] P. X. Li, T. C. Kuo and D. J. A. Trotman, *Blowing-up and Whitney (a)-regularity*, Canadian Math. Bull. Vol 32 (4),1989, 482-485.

[12] D. J. A. Trotman and L. C. Wilson, *Stratifications and finite determinacy*, Proc. London Math. Soc. (3), 1999, 334-368.

[13] J. N. Mather, *Stratifications and mappings*, Dynamical Systems (M. M. Peixoto, ed.), Academic Press, 1973, 195-232.

[14] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Anal. Math. 73, 1961, 295-323.

[15] M. Pflaum, *Analytic and geometric study of stratified spaces*, Lecture Notes in Mathematics, 1768, Springer-Verlag, Berlin, 2001.

[16] R. Sjamaar, *Singular Orbit Spaces in Riemannian and symplectic Geometry*, Ph.D. thesis, Rijksuniversiteit te Utrecht, 1990.

[17] D. J. A. Trotman, *Comparing regularity conditions on stratifications*, Proceedings of Symposia in Pure Mathematics, Volume 40, Arcata 1981-Singularities, Part 2, American Mathematical Society, Providence, Rhode Island, 1983, 575-586.

[18] D. J. A. Trotman, *Counter exemples in stratification theory: two discordant horns*, Real and Complex Singularities, Oslo 1976 (P. Holm, Editor), Stijthoff and Noordhoff, Alpen aan den Rijn, 1977, pp.679-686.

[19] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. 36, 1976, 295-312.